

MUTATIONS AND POINTING FOR BRAUER TREE ALGEBRAS

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Abstract

Brauer tree algebras are important and fundamental blocks in the modular representation theory of groups. In this research, we present a combination of two main approaches to the tilting theory of Brauer tree algebras.

The first approach is the theory initiated by Rickard, providing a direct link between the ordinary Brauer tree algebra and a particular algebra called the Brauer star algebra. This approach was continued by Schaps-Zakay with their theory of pointing the tree.

The second approach is the theory developed by Aihara, relating to the sequence of mutations from the ordinary Brauer tree algebra to the star-algebra of the Brauer tree. Our main purpose in this research is to combine these two approaches:

We find an algorithm based on centers which are all terminal edges, for which we are able to obtain a tilting complex constructed from irreducible complexes of length two [SZ1], which is obtained from a sequence of mutations. For the algorithm given by Aihara in [Ai], which he showed in Cor. 2.6 gives a two term tree-to-star complex, we prove that Aihara's complex is obtained from the corresponding completely folded Rickard tree-to-star complex by a permutation of projectives.

1 INTRODUCTION

This work concerns Brauer tree algebras, a widely studied class of algebras of finite representation type which includes all blocks of cyclic defect group in modular group representation theory. In the last twenty-five years modular group representation theory has been considerably enriched by the introduction of methods from algebra representation theory, foremost among them the theory of tilting complexes in [R1], which led to the Broué conjecture

that every block whose defect group is abelian is derived equivalent to its Green correspondent.

A block of cyclic defect group is a Brauer tree algebra and its Green correspondent is a Brauer star algebra. Rickard proved [R2] that every Brauer tree algebra has a tilting complex which makes it derived equivalent to the corresponding Brauer star algebra. Schaps-Zakay showed that the tilting complexes in the opposite direction can be constructed from irreducible projective complexes of length two. Since all the projectives of the Brauer star algebra have the same simple form, the endomorphism algebra of such a tilting complex is easily constructed and the structure of the Brauer tree can be read off from the complex, making them an excellent introduction to tilting theory.

There have been two main approaches to the tilting theory of Brauer tree algebras: the all-at-once approach, going back to Rickard [R1] (later involving pointing), and the step-by-step approach going back to König and Zimmermann [KZ1], later formulated in terms of mutations by Aihara [Ai] and used recently by Chan [Ch] and Zvonarevna [Zv]. In this project we propose to combine and compare the two approaches.

2 DEFINITIONS AND NOTATION

2.1 DERIVED EQUIVALENCE

We fix an abelian category \mathcal{C} and we denote by $Ch(\mathcal{C})$ the category of cochain complexes of objects of \mathcal{C} . The differentials of the complex are morphisms $\{d_n : C_n \rightarrow C_{n+1}\}$ satisfying $d_{n+1} \circ d_n = 0$. For any given complex C^\bullet , the cocycles $Z^n(C^\bullet)$, coboundaries $B^n(C^\bullet)$, and cohomology modules $H^n(C^\bullet) = Z^n(C^\bullet)/B^n(C^\bullet)$ are defined as in such standard texts as [W].

If C^\bullet, D^\bullet are cochain complexes, a morphism $f_\bullet : C^\bullet \rightarrow D^\bullet$ is a cochain map, which is to say, a family of morphisms $f_n : C_n \rightarrow D_n$ which commute with d in the sense that $f_{n+1}d_n = d_nf_n$. To avoid sounding pedantic, we will usually refer to these simply as chain maps, even when they are mapping cochain complexes.

A morphism $f_\bullet : C^\bullet \rightarrow D^\bullet$ between cochain complexes sends coboundaries to coboundaries and cocycles to cocycles. Thus, it induces module morphisms $f_n^* : H^n(C^\bullet) \rightarrow H^n(D^\bullet)$.

Definition 2.1. A morphism $f_\bullet : C^\bullet \rightarrow D^\bullet$ of cochain complexes is called a *quasi-isomorphism* if the induced maps $H^n(C^\bullet) \rightarrow H^n(D^\bullet)$ are all isomorphisms.

Definition 2.2. A cochain complex C is called *bounded* if almost all the C^n are zero. The complex C is *bounded below* if there is a bound a such that $C^n = 0$ for all $n < a$. The cochain complexes which are partially or fully bounded form full subcategories Ch^b, Ch^+, Ch^- of $Ch(C)$.

Definition 2.3. The *derived category* of an abelian category \mathcal{C} is the category obtained from $Ch(\mathcal{C})$ by adding formal inverses to all the quasi-isomorphisms between chain complexes. It is called the *bounded derived category* and denoted $D^b(\mathcal{C})$ if we consider only bounded complexes. A *derived equivalence* between two abelian categories is an equivalence of categories between their derived categories.

Consider a ring R which is assumed to be associative but not necessarily commutative, which in the sequel will typically be either a block of a group algebra over a field of characteristic p dividing the order of the group or else a finite dimensional algebra over a field of arbitrary characteristic. The category $R - Mod$ of left R -modules is the abelian category of primary interest to us. For any such ring R , let $D^b(R)$ be the derived category of bounded complexes of left R -modules.

Definition 2.4. We say that a cochain map $f : C \rightarrow D$ is *null homotopic* if there are maps $s_n : C^n \rightarrow D^{n+1}$ such that $f = ds + sd$.

Definition 2.5. Two cochain maps $f, g : C \rightarrow D$ are *chain homotopic* if their difference $f - g$ is null homotopic, in other words, if $f - g = sd + ds$ for some s . The maps $\{s_n\}$ are called a *homotopy* from f to g . We say that $f : C \rightarrow D$ is a *homotopy equivalence* if there is a map $g : D \rightarrow C$ such that $g \circ f$ is chain homotopic to the identity map id_C and $f \circ g$ is chain homotopic to the identity map id_D .

Definition 2.6. Let $f : C \rightarrow D$ be a map of cochain complexes. The *mapping cone* of f is a chain complex $Cone(f)$ whose degree n part is $C_{n+1} \oplus D_n$. The differential in $Cone(f)$ is given by the formula:

$$d(c, b) = (-d_C(c), d_D(b) + f(c)), \quad c \in C_{n+1}, \quad b \in D_n$$

Any map $H^n(f) : H^n(C) \rightarrow H^n(D)$ can be fit into a long exact sequence of cohomology groups by use of the following device. There is a short exact sequence

$$0 \rightarrow D \rightarrow \text{Cone}(f) \xrightarrow{\delta} C[1] \rightarrow 0$$

of cochain complexes, where the left map sends b to $(0, b)$, and the right map sends (c, b) to $-c$.

2.2 TILTING COMPLEXES

We now consider complexes T of projective modules over an associative ring R . The notation $T[n]$ denotes the complex which is isomorphic to T as a module but in which the gradation has been shifted n places to the left and the differential is the shift of the differential multiplied by $(-1)^n$.

For any ring R , let $D^b(R)$ be the derived category of bounded complexes of R -modules.

Definition 2.7. Let R be a Noetherian ring. A bounded complex T of finitely generated projective R -modules is called a *tilting complex* if:

- (i) $\text{Hom}_{D^b(R)}(T, T[n]) = 0$ whenever $n \neq 0$.
- (ii) For any indecomposable projective P , define the stalk complex to be the complex $P^\bullet : 0 \rightarrow P \rightarrow 0$. Then every such P^\bullet is in the triangulated category generated by the direct summands of direct sums of copies of T .

A complex T satisfying only (i) is called a *partial tilting complex*.

Definition 2.8. Fix an abelian category \mathcal{C} and the category of cochain complexes $Ch(\mathcal{C})$. For two complexes X and Y denote by $Z(X, Y)$ the set of morphisms from X to Y which are homotopic to zero. The collection of all $Z(X, Y)$ forms a subgroup of $\text{Hom}_{Ch(\mathcal{C})}(X, Y)$. Denote by $K(\mathcal{C})$ the quotient category, i.e. $K(\mathcal{C})$ is the category having the same objects as $Ch(\mathcal{C})$ but with morphisms

$$\text{Hom}_{K(\mathcal{C})}(X, Y) = \text{Hom}_{Ch(\mathcal{C})}(X, Y) / Z(X, Y),$$

so that two homotopic maps are identified. The quotient category $K(\mathcal{C})$ is called the *homotopy category*, and a homotopy equivalence between complexes is an isomorphism in the homotopy category. For an abelian category \mathcal{C} , $K^-(\mathcal{C})$ is the homotopy category of right bounded complexes in \mathcal{C} , and similarly one can define $K^+(\mathcal{C})$.

The derived category is not an abelian category, but it is a triangulated category. The original theory of tilting concerned modules called tilting modules. Happel, in [H] showed that if there was a tilting between two algebras Λ and Γ , it induced a functor which was an equivalence of their derived bounded categories. Rickard [R1] then proved a converse when tilting modules were replaced by tilting complexes, namely, that there is a tilting complex T over Λ with endomorphism ring $\text{End}_{D^b(\Lambda)}(T)^{op} \cong \Gamma$.

2.3 BRAUER TREES

From now on, we concentrate on a particular class of algebras, the Brauer tree algebras. Even when the algebra which interests us is the block of a group algebra, we will not use the actual block but rather its skeleton, a Morita equivalent algebra which is basic, so that the quotient by the radical is a direct sum of copies of the field. Suppose that in the original block, the dimension of the i th simple module was m_i . When we have finished calculating the tilting complex $T = \bigoplus T_i$ using the skeleton, then we can recover the original block as opposite algebra of the endomorphism ring of the tilting complex $T' = \bigoplus T_i^{\oplus m_i}$.

Definition 2.9. Let e and m be natural numbers. A *Brauer tree* of type (e, m) is a finite tree (V, \mathcal{E}) where V is the set of vertices, \mathcal{E} is the set of edges, $|\mathcal{E}| = e$ (hence $|V| = e + 1$), together with a cyclic ordering of the edges at each vertex and a designation of an exceptional vertex which is assigned multiplicity m .

The set of all edges of vertex u is denoted by $\mathcal{E}(u)$. By "cyclic ordering" we mean that for each edge E in $\mathcal{E}(u)$ there is a 'next' edge in $\mathcal{E}(u)$ and that edge has a next edge in $\mathcal{E}(u)$ etc., until each edge of u is counted exactly once, in which case E is the next one. We note that if E and F are the only edges of u then F is next after E and E is next after F .

Every Brauer tree can be embedded in the plane in such a way that the cyclic ordering on each $\mathcal{E}(u)$ is the counterclockwise direction. The exceptional vertex will be drawn as a black circle and the other vertices as open circles. Two important examples of Brauer trees are:

- (i) The *star* with the exceptional vertex in the center.

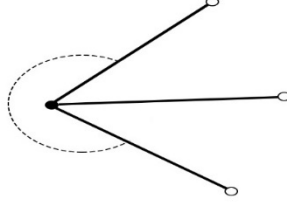


Figure 1

(ii) The *linear tree*, which includes, for example, the Brauer trees of blocks of cyclic defect in the symmetric groups.

We relate Brauer trees to the structure of algebras. In the definition we need to refer to uniserial modules, these being modules in whose radical series each submodule has a simple top, where the top of a module is the quotient by the radical.

Definition 2.10. An algebra A is called a *Brauer tree algebra* if there is a Brauer tree such that the indecomposable projective A -modules can be described by the following algorithm:

- (i) There is bijection between the edges of the tree and the isomorphism classes of simple A -modules, i.e. each edge is labelled by the corresponding isomorphism class.
- (ii) If S is a simple A -module and P_S is the corresponding indecomposable projective A -module then $P_S \supseteq \text{rad}(P_S) \supseteq \text{soc}(P_S) \cong S$ and $\text{rad}(P_S)/\text{soc}(P_S)$ is a direct sum of one or two uniserial modules corresponding to the two vertices of the edge, with composition factors determined by a clockwise circuit around the vertex.

Definition 2.11. Let e and m be natural numbers with $e > 1$. Let K be any field containing a primitive e th root of unity ξ . Let $\hat{n} = em + 1$. Let the cyclic group $C_e = \langle g \rangle$ act on the truncated polynomial ring $A = K[x]/x^{\hat{n}}$, $g : x \mapsto \xi x$. The *Brauer star algebra* of type (e, m) is the skew group algebra $b = A[C_e]$, in which g and x obey the relation $g^{-1}xg = \xi x$. The algebra b has e distinct simple modules, corresponding to the idempotents

$$f_i = \frac{1}{e} \sum_{j=0}^{e-1} \xi^{-ij} g^j, \quad i = 1, \dots, e,$$

and satisfying $f_i x = x f_{i+1}$.

The corresponding indecomposable projective left modules are denoted by $P_i = b f_i$, $i = 0, \dots, e-1$. Each P_i is uniserial, and the projective cover of $\text{rad}(P_{i+1})$ is P_i . We let $\{x^s f_i\}_{s=0}^{em}$ be a basis for P_i , and define the following maps:

$$\varepsilon_i : P_i \rightarrow P_i, \quad \varepsilon_i(f_i) = x^e f_i$$

$$\tilde{h}_{ij} : P_i \rightarrow P_j, \quad \tilde{h}_{ij}(f_i) = x^k f_j, \quad k \equiv j - i \pmod{e}, \quad 0 \leq k < e.$$

For $i \neq j$, we denote \tilde{h}_{ij} by h_{ij} , and for $i = j$ by id_i . For any $0 \leq \ell \leq m$ we call a map $\varepsilon_j^\ell \tilde{h}_{ij} (= \tilde{h}_{ij} \varepsilon_i^\ell)$ *normal homogeneous* of degree $\ell e + k$, where

$$k \equiv j - i \pmod{e}, \quad 0 \leq k < e.$$

Corollary ([SZ1, Lemma 1.1]). *For the Brauer star algebra b , if $j \neq i$, and $\{j - i\}_e$ is the residue mod e , then there are m normal homogeneous maps $\varepsilon^\ell h_{ij}$, with degrees s for $s = \{j - i\}_e + \ell e$ where $\ell = 0, \dots, m-1$. If $j = i$, then there are $m+1$ normal homogeneous maps $\varepsilon^s : P_i \rightarrow P_i$, for $s = 0, e, 2e, \dots, me$.*

Definition 2.12. A cochain map l_\bullet between C^\bullet and D^\bullet is called *normal homogeneous* if each vertical map is normal homogeneous.

Definition 2.13. We call the homomorphism $\varepsilon_i^m : P_i \rightarrow P_i$ the *socle map*, for the obvious reason that it maps the top of P_i into its socle $\langle x^{em} f_i \rangle$.

A partial tilting complex T for the Brauer star algebra b is called *two-restricted* (PTC_2) if it is a direct sum of shifts of the indecomposable complexes

$$\begin{array}{lcl} S_i : & 0 & \rightarrow P_i \rightarrow 0 \\ T_{ij} : & 0 & \rightarrow P_i \rightarrow P_j \rightarrow 0, \quad i \neq j \end{array}$$

where the first nonzero component of S_i and T_{ij} is in degree zero. The complexes $S_i[n]$ and $T_{ij}[n]$ are called *elementary*. The map from T_{ij} to T_{ij} which is ε_i^m on P_i and zero on P_j is called the *socle chain map*. It is chain homotopy equivalent to the map which is zero on P_i and $-\varepsilon_j^m$ on P_j . Note

that one can show that any indecomposable complex satisfying Def 2.7(i), which is nonzero in at most two degrees, is elementary and that a basis of the endomorphism ring of a tilting complex in TC_2 is given by the normal homogeneous maps [SZ1]

Definition 2.14. Let B be a Brauer tree of type (e, m) . A *pointing* on B is the choice, for each nonexceptional vertex u , of a pair of edges (i, j) which are adjacent in the cyclic ordering at u . If there is only one edge i at u , then we take (i, i) as the required pair. The tree B together with a pointing is called a *pointed Brauer tree*.

Remark 2.1. Recall that we have represented each Brauer tree by a planar embedding and the cyclic ordering at each vertex by counterclockwise ordering of the edges in the plane. We then represent the pointing (i, j) by placing a point in the sector between edge i and edge j in a small neighborhood of u , as in Figure 2.

Definition 2.15. Let B be a Brauer tree with vertex set V . The distance $d(u)$ of any vertex $u \in V$ from the exceptional vertex u_0 is the number of edges in a minimal path from u to u_0 (and hence in any path without backtracking, since the graph is acyclic).

Definition 2.16. Let B be a Brauer tree with edge set \mathcal{E} . An *edge numbering* of B is a Brauer tree with all its edges numbered by $1, \dots, e$. The *vertex numbering* of B is obtained from the edge numbering by giving the same number as the edge to the farthest vertex from the exceptional vertex on the edge. The exceptional vertex is numbered as 0.

Definition 2.17. A *Green's walk* for a planar tree is a counterclockwise circuit of the tree as if one were walking around the tree touching each edge with the left hand. Each pointing and each choice of an initial branch determines an edge numbering by starting at the exceptional vertex v and taking a Green's walk around the tree which begins with the initial branch, and numbering the vertices and corresponding edges as $1, 2, 3, \dots, e$ as one come to the points. Figure 2 shows an example of such a vertex pointing.

Definition 2.18. At any vertex besides the exceptional vertex, we will call the first edge that one would meet on a Green's walk around the tree the *primary edge* of the vertex, and the first edge one would meet on a reversed Green's walk will be called the *coprimary edge*. The pointing which puts the point between the entering edge and the primary edge at each vertex will be called the *ordinary pointing* and the pointing which puts the point between the entering vertex and the coprimary edge will be called the *reversed pointing*. The pointing which places the point first to the left and then to the right of the entering vertex, alternating as one goes out from the exceptional vertex, will be called the *left alternating pointing*, and there is dual which we will not need.

As described in [SZ2], each pointing determines a two-restricted star-to-tree tilting complex, in which the projectives of the tilting complex are from the Brauer star with the same (e, m) and the opposite algebra of the endomorphism ring in the homotopy category is isomorphic to the Brauer tree algebra of the tree which was pointed. The components T_i of this star-to-tree complex are stalk complexes for edges at the exceptional vertex and complexes $T_{ij}[-n_i]$ or $T_{ji}[-n_i + 1]$, depending on whether the point is before or after i in the cyclic ordering from the entering vertex j . The shifts are adjusted so that every P_i appears in a unique degree n_i . A different pointing would give a different tilting complex with isomorphic endomorphism ring. In the example in Figure 2, the vertex numbered 7 corresponds to a partial two-restricted tilting complex, ordered so that the vertical maps are generators of the endomorphism ring of the partial tilting complex. Note that the socle map is located just where the point is in the diagram, between 5 and 8.

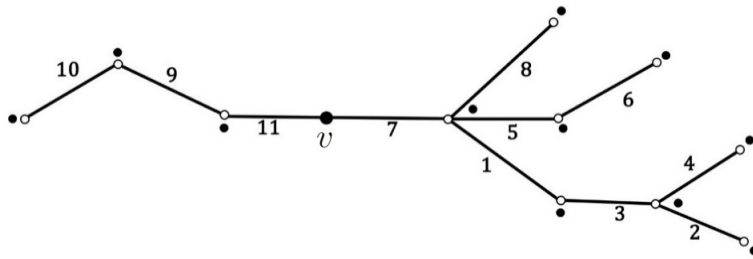


Figure 2

$$\begin{array}{ccccccc}
T_7 : & & 0 & \rightarrow & P_7 & \rightarrow & 0 \\
& & & & id \downarrow & & \\
T_1 : & 0 & \rightarrow & P_1 & \rightarrow & P_7 & \rightarrow 0 \\
& & & h_{15} \downarrow & & id \downarrow & \\
T_5 : & 0 & \rightarrow & P_5 & \rightarrow & P_7 & \rightarrow 0 \\
& & & & & \varepsilon^m \downarrow & \\
T_8 : & & 0 & \rightarrow & P_7 & \rightarrow & P_8 \rightarrow 0 \\
& & & & id \downarrow & &
\end{array}$$

Definition 2.19. Consider a sequence $\{r_i\}_{i=1}^l$ of elements of $\{1, \dots, e\}$. Set

$$h = \tilde{h}_{r_{l-1}r_l} \circ \dots \circ \tilde{h}_{r_1r_2} = \varepsilon_{r_l}^\alpha \tilde{h}_{r_1r_l}.$$

Then the sequence is *short* if $\alpha = 0$ and *long* if $\alpha > 0$. We generally represent the sequence in the form $r_1 \rightarrow r_2 \rightarrow \dots \rightarrow r_l$.

Example 1. If $e = 11$ as in the example above,

- $1 \rightarrow 5 \rightarrow 7$ is short
- $5 \rightarrow 1 \rightarrow 7$ is long.

There is a result from [SZ1] showing that a chain map $\ell_\bullet : T_{ik} \rightarrow T_{jk}$ has the identity map at P_k if $i \rightarrow j \rightarrow k$ is short and is the socle map if $i \rightarrow j \rightarrow k$ is long, and similarly for the dual map from T_{ij} to T_{ik} .

2.4 MUTATION

It is, of course, possible to define tilting complexes between two general Brauer tree algebras. Of particular importance are the tilting mutations of [Ai], which go back to work of Rickard [R2] and Okuyama [O], or alternatively, to Kauer [K]. Let A be a finite dimensional basic algebra, with projective modules P_j . To each j , we can associate an idempotent \tilde{e}_j with

$$1_A = \sum_{j=1}^e \tilde{e}_j.$$

Definition 2.20. Fix an i and define $e_0 = \sum_{j \neq i} \tilde{e}_j$. For any $j \in \mathcal{E}$ we define a complex by

$$T_j^{(i)} = \begin{cases} (0th) & (1st) \\ P_j & \longrightarrow 0 & j \neq i \\ Q_i & \xrightarrow{\pi_i} P_i & j = i \end{cases}$$

where $Q_i \xrightarrow{\pi_i} P_i$ is a minimal projective presentation of $\tilde{e}_j A / \tilde{e}_j A e_0 A$. Now we define $T^{(i)} := \oplus_{j \in \mathcal{E}} T_j^{(i)}$. The *mutation* μ_i^+ of A is $A' \cong \text{End}_{D^b(A)} T^{(i)}$. We will also consider the dual variant, as in [S].

$$T_j^{(i)} = \begin{cases} (-1st) & (0th) \\ 0 & \longrightarrow P_j & j \neq i \\ P_i & \xrightarrow{\pi_i} Q_i & j = i \end{cases}$$

where Q_i is the minimal injective hull of the quotient of P_i by the largest submodule containing only components isomorphic to the simple module S_i . This injective hull will be a direct sum of injective modules (which are also projective) whose irreducible socles give the socles of this quotient. We will denote this by $\mu^-[\text{AI}]$ (see, e.g., [Zv] for more detail in the case of Brauer trees.)

Since A is a symmetric algebra, either version of the mutation will give a tilting complex. (A similar complex can be defined also if A is not symmetric, but then we get a complex which is no longer a tilting complex.)

Now let A be a Brauer tree algebra. Aihara showed in [Ai] that there is a simple combinatorial operation on edges $j \in \mathcal{E}$ which corresponds to the mutation: The edge j is detached from both of its endpoints, and reattached to the tree at the farther end of the edge which is next before it in cyclic ordering. If the edge j is a leaf, then there is only one reattachment made.

By dualizing of Aihara's main theorem, [Ai] Theorem 2.2, the mutation μ^- would correspond to the dual version of Aihara's operation on the Brauer tree, namely, reattaching to the farther end of the edge which is *after* it in the cyclic ordering. The diagrams to demonstrate this can be found in [Zv].

Lemma 2.1. *The inverse functor to the functor G^+ given by a mutation μ_i^+ is the functor G^- given by μ_i^- .*

Proof. Let us assume that all the stalk complexes are in degree 0. We let the projectives in the Brauer tree algebra A'' on which μ_i^+ is acting be denoted by P_j'' , and the corresponding projective modules in the algebra A' on which

μ_i^- is acting be denoted by P_j' . Let Q_i'' be the corresponding projective cover of the radical, and let Q_i' be the corresponding injective hull of the socle quotient. Because of the biserial property of projectives of Brauer tree algebras, module Q_i'' is the sum of one or two projectives, corresponding to the edges to which i is reattached, and similarly the projective-injective Q_i' is the direct sum of the corresponding two projectives. The actions of our two functors on the projectives are given by

$$\begin{aligned} G^+ \left(P_j' \right) : & \quad 0 \rightarrow P_j'' \rightarrow 0, j \neq i \\ G^+ \left(P_i' \right) : & \quad 0 \rightarrow Q_i'' \rightarrow P_i'' \rightarrow 0 \\ G^- \left(P_j'' \right) : & \quad 0 \rightarrow P_j' \rightarrow 0, j \neq i \\ G^- \left(P_i'' \right) : & \quad 0 \rightarrow P_i' \rightarrow Q_i' \rightarrow 0 \end{aligned}$$

Since the projectives correspond for all $j \neq i$, all we need to show is that that the cone $Cone \left(G^+ \left(P_i' \right) \xrightarrow{l_0} G^+ \left(Q_i' \right) \right)$, with l_0 given by identity maps between the two copies of Q_i' , is homotopy equivalent to P_i'' .

First, we define chain maps f_\bullet from $Cone(l_\bullet)$ to P_i'' and g_\bullet from P_i'' to $Cone(l_\bullet)$.

$$f_{-1} = 0$$

$$f_0 = (\pi_{P_i''})$$

$$g_{-1} = 0$$

$$g_0 = (0, id_{P_i''})$$

The resulting diagram is as follows,

$$\begin{array}{ccc} Q_i'' & \xrightarrow{(id_{Q_i''}, h'')} & Q_i'' \oplus P_i'' \\ \downarrow 0 & & \downarrow -h'' \circ \pi_{Q_i''} + \pi_{P_i''} \\ 0 & \xrightarrow{0} & P_i'' \\ \downarrow 0 & & \downarrow (0, id_{P_i''}) \\ Q_i'' & \xrightarrow{(id_{Q_i''}, h'')} & Q_i'' \oplus P_i'' \end{array}$$

and a diagram chase will show that both f_\bullet and g_\bullet are chain maps. The composition from the stalk complex to itself is the identity. The homotopy from the composition $h_\bullet = g_\bullet \circ f_\bullet$ to the identity is given by $T = (-\pi_Q)$.

Using functoriality, we pull the functor G^+ outside the cone. Hitting it on the left by $(G^+)^{-1}$ we get

$$(G^+)^{-1}(P_i'') = \left(\text{Cone} \left(P_i' \rightarrow Q_i' \right) \right) = G^-(P_i'')$$

and the remaining stalk complexes all correspond, which gives the desired result. \square

In the same paper [Ai], in Corollary 2.6, Aihara gave an algorithm for reducing a Brauer tree to a Brauer star. In essence, the algorithm consists in doing a mutation centered at an edge of distance one from the exceptional vertex, as long as such edges exist. In terms of number of steps to the star, this class of algorithms is very efficient, requiring only $e - \ell$ steps, where ℓ is the number of branches at the exceptional vertex, since each step creates a new branch at the exceptional vertex.

Any mutation on a symmetric algebra gives a tilting and produces another symmetric algebra. Thus if we have a sequence of mutations, we get a derived equivalence and hence a tilting complex. Furthermore, any mutation of Brauer trees produces a one-to-one correspondence of edges. Thus, if we have a sequence of mutations leading to the Brauer star, any of the natural counterclockwise numberings of the Brauer star will induce a numbering of the Brauer tree. The subject of this paper is the relationship between the choice of reduction procedure and this natural numbering.

3 MUTATION REDUCTION

Assume we are given a Brauer tree G , with multiplicity m . If $m > 1$, then there is a designated exceptional vertex v . For $m = 1$, we assume that one of the vertices has been chosen as the exceptional vertex v . Since our graph is a tree, there is a well-defined distance of each vertex u from v given by counting the number of edges on the unique path connecting them. If the edges of the tree are labelled, then each vertex can be given the same label as the first edge on this unique path.

Definition 3.1. A *mutation reduction* is a mutation or sequence of mutations such that the distance of each vertex from the exceptional vertex does not ever increase, and such that at least one such distance actually decreases. A mutation reduction which ends at the Brauer star is called *complete*.

Lemma 3.1. *Assume we are given a Brauer tree.*

1. *A mutation which is a mutation reduction must be centered at a primary edge.*
2. *A mutation centered at a primary edge connected to an edge adjacent to the exceptional vertex is a mutation reduction.*
3. *After a complete mutation reduction, all the edges from a given branch form an interval around the Brauer star, and these intervals follow the counterclockwise ordering of the branches.*

Proof. 1. If the mutation is not centered at a primary edge, then the mutation reattaches the center at the far end of the edge before it in the cyclic ordering, which is at greater distance from the exceptional vertex, in contradiction to our assumption that we have a mutation reduction.

2. The only mutation which can change the branch structure under a mutation reduction is a mutation by a primary edge w connected to an edge u adjacent to the exceptional vertex. The effect of such a mutation is to create a new branch by lopping off w and the subgraph S of all edges connected to the exceptional vertex through the center w of the mutation. The original branch rooted at u will now be replaced by two branches, one rooted at w and connected to S at the vertex at the opposite end of the edge t which was last in counterclockwise order at the vertex to which w was originally connected. The other branch will be rooted at u , will be changed only in that w and S were removed, and will follow the branch rooted at w immediately in the counterclockwise ordering at v .

It remains to show that this operation was actually a mutation reduction. The edge w , once at distance 1, is now at distance 0. The edge t , once at distance 2, is now at distance 1, and every edge originally connected to t and thus connected to the exceptional vertex via three edges, u, w, t , is now connected to v via w and is therefore at distance two less than before. Finally, the remaining edges of S , now all connected to the exceptional vertex via w, t instead of u, w , remain at exactly the same distance that they had before.

3. In any complete mutation reduction, each branch is eventually split entirely into separate leaves attached to the exceptional vertex. However, since this is always done, as described above, by separating one branch into two adjacent branches with the same labels as the original branch, the end result is that all the edges of in the original branch correspond to an interval around the star.

□

We will examine two different mutation reduction algorithms, one a version of the original algorithm given by Aihara [Ai] and the other our own from [Z].

Aihara's Algorithm[Ai]

1. Choose an initial branch.
2. In a Green's walk starting at the root of the the initial branch choose the first primary edge w attached to an edge adjacent to the exceptional vertex. If the tree is not a star, there must be such an edge.
3. By Lemma 3.1(1), the mutation centered on this edge w is a mutation reduction, and from the proof we see that it creates two adjacent branches from the original, the first of which in counter-clockwise order is rooted at w .
4. If w was on the initial branch, let the new initial branch be the new branch rooted at w , and otherwise let the initial branch remain as before. Begin again from 2.

Algorithm Z[Z]

1. Choose an initial branch. Let $d > 1$ be the maximal distance of a vertex from the exceptional vertex v .
2. In a Green's walk starting on the initial branch choose the first leaf at distance d , necessarily a primary edge, as center, and perform a series of mutations centered on the edge with this label, for as long as it remains a primary edge or until it reached the exceptional vertex. The distances of all other vertices from the exceptional vertex will be unchanged.
3. Choose the next leaf at distance d and proceed as in the previous item.

4. When there are no leaves left at distance d , then we find the new maximal distance d' . If $d' = 1$ we are finished, and otherwise we set $d = d'$ begin again at 2.

In terms of number of steps, this is as inefficient as a mutation reduction algorithm can be, because at each step, only one edge has its distance reduced by one.

We now construct a numbering on a Brauer tree depending on which algorithm we use, by starting with the interval coming from the chosen initial branch and numbering the edges in order. This numbering will be called the *natural numbering* corresponding to this choice of initial branch.

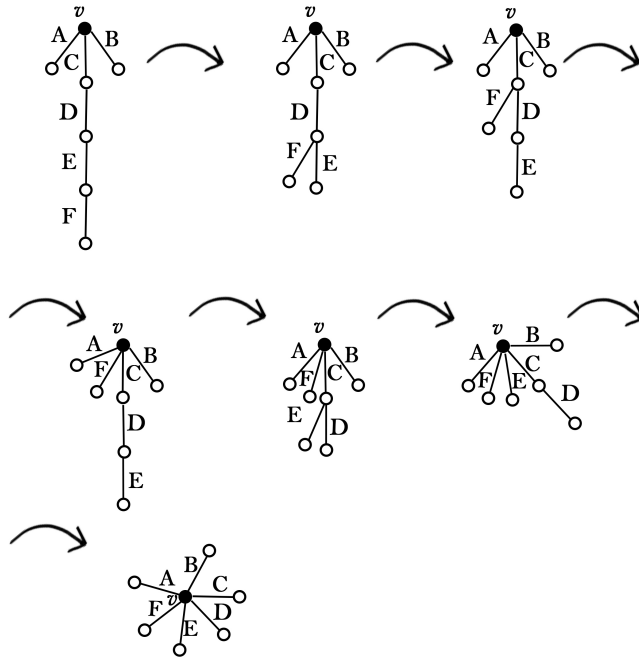


Figure 3

Example 2. In Figure 3, we got to the Brauer star using Algorithm Z. Taking the largest branch as initial branch, we get as natural numbering

$$A = 6, B = 5, C = 4, D = 3, E = 2, F = 1$$

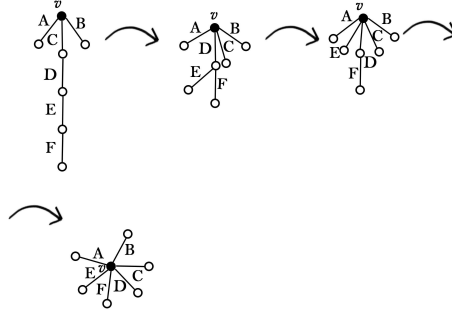


Figure 4

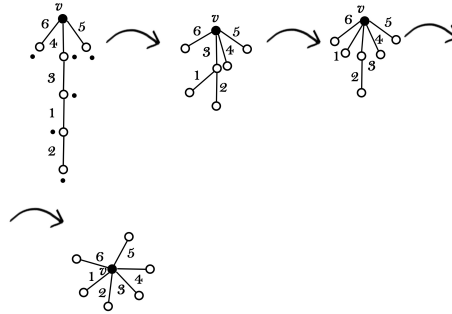


Figure 5

Example 3. Now, we follow Aihara's Algorithm for the same original Brauer tree. In Figure 4, using the same initial branch, we get a different natural numbering.

$$A = 6, B = 5, C = 4, D = 3, E = 1, F = 2$$

The numbering gives a pointing and this pointing gives us a corresponding star-to-tree tilting complex as described in §2. We will prove that from Algorithm Z we get a tilting complex corresponding to the natural numbering. From Aihara's Algorithm we obtain a tilting complex which comes from a pointing but we will show that it is not usually the pointing corresponding to the natural numbering.

4 MAIN THEOREM

The basic step in Algorithm Z is to take a leaf C which is a primary edge in a Brauer tree giving an algebra A'' and to do a mutation centered on this leaf, which will be a mutation reduction to a tree whose algebra is A' . The tilting complex of this mutation is expressed in terms of the projectives P_i'' of A'' , given by a functor $G : D^b(A') \rightarrow D^b(A'')$. Since C is a primary edge attached to some edge B which is closer to the exceptional vertex, the functor G will act as the identity for every projective P_i' of A' except P_C' , and for C itself we will have the projective cover of its radical, which is P_B' because C is a primary edge adjacent to B .

$$\begin{array}{lcl} G(P_i') : & 0 \rightarrow P_i'' \rightarrow 0 \\ G(P_C') : & 0 \rightarrow P_B'' \rightarrow P_C'' \rightarrow 0 \end{array}$$

Theorem 4.1. *For any complete mutation reduction whose centers are always leaves which are primary edges, the star-to-tree tilting complex of the composed mutations is the star-to-tree complex of the original tree with the reversed pointing.*

Proof. Let $A_\ell, A_{\ell-1}, \dots, A_1$ be the Brauer tree algebras in the complete mutation reduction to the Brauer star algebra A_0 . We number each of the corresponding Brauer trees by the natural numbering corresponding to this mutation reduction. For each k between 1 and ℓ , we let

$$F_k : D^b(A_0) \rightarrow D^b(A_k)$$

$$F_k^{-1} : D^b(A_k) \rightarrow D^b(A_0)$$

be the functors obtained by composing the functors G^+ of the mutations μ^+ and, respectively, the functors G^- of the dual mutations μ^- in the opposite order. We want to show that the star-to-tree tilting complex given by F_ℓ^{-1} is the star-to-tree complex given by the reversed pointing, from which it will follow by the results of [RS] that the tilting complex inducing F_ℓ is Rickard's tree-to-star complex for the same pointing.

Let us prove the theorem by induction on ℓ . If ℓ is 1, then the Brauer tree has only one edge w not attached to the exceptional vertex v , but rather to some u attached to v . If w is numbered i after mutation, then u will be numbered by $i + 1$ since it comes after the new w in the cyclic ordering of

the star. In the tilting complex of μ_i^- we will have $Q_i = P_{i+1}$, and thus it coincides with the star-to-tree complex of the reversed pointing.

Now assume that the theorem is true for $\ell - 1$, so that $F_{\ell-1}^{-1}$ gives the star-to-tree complex of the reversed pointing of the Brauer tree. Let the $\{P'_j\}$ be the projective left modules of $A_{\ell-1}$ and let the $\{P''_j\}$ be the projective left modules of A_ℓ . Let i be the number of the center of the mutation in the Brauer tree of $A_{\ell-1}$ and let j be the number of the next edge after it in the cyclic ordering. By the rules for numbering edges at a vertex, we must have $j > i$. Furthermore, since i is a leaf, P'_i is uniserial, so the injective hull Q'_i of $P'_i/\text{Soc}(P'_i)$ is P'_j .

$$\begin{array}{lcl} G^-(P''_j) : & 0 & \rightarrow P'_j \rightarrow 0, j \neq i \\ G^-(P''_i) : & 0 & \rightarrow P'_i \rightarrow P'_j \rightarrow 0 \end{array}$$

Case 1. The edge j is not attached to the exceptional vertex:

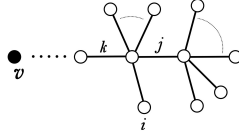


Figure 6, $A_{\ell-1}$

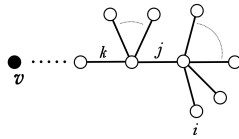


Figure 7, A_ℓ

Let k be the entering edge of the vertex at which i and j meet in the Brauer tree of $A_{\ell-1}$, as in Figure 6, and assume that it is in degree n_k in the tilting complex. Then by the assumption of reversed pointing, we have $i < j < k$, and thus $F_{\ell-1}^{-1}(P'_i) = T_{ik}[n_k + 1]$ and $F_{\ell-1}^{-1}(P'_j) = T_{jk}[n_k + 1]$. In this case we

will get that the composition

$$\begin{aligned}
F_\ell^{-1}(P_i'') &= F_{\ell-1}^{-1} \circ G^{-1}(P_i'') \\
&= F_{\ell-1}^{-1}\left(\text{Cone}\left(P_i' \rightarrow P_j'\right)\right) \\
&= \text{Cone}\left(F_{\ell-1}^{-1}(P_i') \rightarrow F_{\ell-1}^{-1}(P_j')\right) \\
&= \text{Cone}\left(\begin{array}{ccccc} 0 & \longrightarrow & P_i & \xrightarrow{h_{ik}} & P_k & \longrightarrow & 0 \\ & & \downarrow h_{ij} & & \downarrow id & & \\ 0 & \longrightarrow & P_j & \xrightarrow{h_{jk}} & P_k & \longrightarrow & 0 \end{array}\right)
\end{aligned}$$

Denote the chain map in the cone by l^\bullet . We compute $\text{Cone}(l^\bullet)$ and get:

$$P_i \xrightarrow{(-h_{ik}, h_{ij})} P_k \oplus P_j \xrightarrow{(\pi_k + h_{jk} \circ \pi_j)} P_k$$

where P_k is in degree 0. We want to show that $\text{Cone}(l^\bullet)$ is homotopy equivalent to $T_{ij}[2]$, which is to say, T_{ij} shifted so that the P_i is in degree -2.

The chain maps f_\bullet and g_\bullet in the following diagram can be checked by composition or by diagram chasing.

$$\begin{array}{ccccc}
P_i & \xrightarrow{(-h_{ik}, h_{ij})} & P_k \oplus P_j & \xrightarrow{(\pi_k + h_{jk} \circ \pi_j)} & P_k \\
\downarrow id & & \downarrow \pi_j & & \downarrow 0 \\
P_i & \xrightarrow{h_{ij}} & P_j & \longrightarrow & 0 \\
\downarrow id & & \downarrow (-h_{jk}, id) & & \downarrow 0 \\
P_i & \xrightarrow{(-h_{ik}, h_{ij})} & P_k \oplus P_j & \xrightarrow{(\pi_k + h_{jk} \circ \pi_j)} & P_k
\end{array}$$

The composition $f_\bullet \circ g_\bullet$ is the identity, so we need only prove that the composition $h_\bullet = g_\bullet \circ f_\bullet$ is homotopic to the identity of the mapping cone. We need to find $T_1 : P_k \oplus P_j \rightarrow P_i$, $T_2 : P_k \rightarrow P_k \oplus P_j$ such that:

$$\begin{array}{ccccc}
P_i & \xrightarrow{(-h_{ik}, h_{ij})} & P_k \oplus P_j & \xrightarrow{(\pi_k + h_{jk} \circ \pi_j)} & P_k \\
id \downarrow \downarrow id & & \downarrow id & & \downarrow 0 \\
& \nearrow T_1 & & \nearrow T_2 & \\
P_i & \xrightarrow{(-h_{ik}, h_{ij})} & P_k \oplus P_j & \xrightarrow{(\pi_k + h_{jk} \circ \pi_j)} & P_k
\end{array}$$

To get the homotopy that we want, we choose $T_1 = 0$ and $T_2 = (-id, 0)$.

Case 2. Near the exceptional vertex: i is adjacent to the exceptional vertex after doing the mutation. In Figure 8 we have the Brauer tree for A_ℓ and in Figure 9 we have the relevant portion of the Brauer tree for $A_{\ell-1}$.

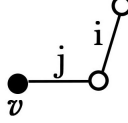


Figure 8

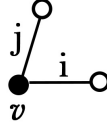


Figure 9

When we compute the tilting complex of the mutation, the component of P_i'' also is two-restricted. In this case we will get that the composition

$$F_{\ell-1}^{-1} \circ G^{-1} (P_i'') = Cone \left(\begin{array}{ccccc} 0 & \longrightarrow & P_i & \longrightarrow & 0 \\ & & \downarrow h_{ij} & & \\ 0 & \longrightarrow & P_j & \longrightarrow & 0 \end{array} \right)$$

which is homotopy equivalent to $T_{ij}[1]$, and equal to $F_\ell^{-1} (P_i'')$, which is precisely what we need for the star-to-tree tilting complex of the reversed pointing.

□

5 AIHARA'S ALGORITHM

In Cor. 2.6 of [Ai], Aihara shows that the tree-to-star functor obtained by composing the mutations is of length two. We will compare Aihara's functor

with the completely folded two-term version of Rickard's tree-to-star functor given in [RS].

Proposition 5.1. *We consider an arbitrary Brauer tree algebra. Let σ be the permutation of $1, \dots, e$ sending each number in the natural numbering of the tree by Aihara's Algorithm to the number of the corresponding edge in the left alternating numbering. Then*

1. *The star-to-tree complex obtained by composing the mutations of the algorithm in reverse order can be obtained from the star-to-tree complex of the left alternating pointing by permuting the rows by σ .*
2. *The tree-to-star complex corresponding the Aihara's Algorithm is the completely folded Rickard tree-to-star complex for the left alternating pointing, except that the projectives are permuted by σ .*

Proof. 1. We let ℓ be the number of mutations in the complete mutation reduction. In the case $\ell = 1$, the center of the mutation is a leaf, so the natural numbering the the reversed numbering, and for a linear tree of length 2, this coincides with the left alternating numbering, so the permutation is the identity. We let F_ℓ^{-1} be the star-to-tree functor obtained by composing the inverse mutations, and let H_ℓ^{-1} be the star-to-tree functor given by the left alternating numbering, with σ_ℓ the permutation mapping the natural number of an edge to its number in the left alternating numbering.

We assume, by induction, that the proposition is true for $\ell - 1$. Let the P'_i be the projectives for $\ell - 1$ in the natural numbering, and let P''_i be the projectives for ℓ in the natural numbering. Let the Q'_i be the projectives for $\ell - 1$ in the left alternating numbering, and let Q''_i be the projectives for ℓ in the left alternating numbering. By our induction hypothesis, this means that for every $i, 1 \leq i \leq e$,

$$F_{\ell-1}^{-1}(P'_i) = H_{\ell-1}^{-1}(Q'_{\sigma_{\ell-1}(i)})$$

For any branch, let the edge connected to the exceptional vertex be called the root. We are now going to perform an inverse mutation centered at a root w , which will join the branch with root w to the next branch, with root u , where, as stated in Lemma 3.1(3), we have $u > w$. Let $B_{\ell-1}$ be the Brauer tree before the branches rooted at u and w are joined, and let B_ℓ be the Brauer tree after they are joined. We let t be the primary edge connected to w , and the same result

shows that w is the numerically highest number in the branch, so that $w > t$. We note that for the left alternating numbering the number of the root is also the highest in the branch, so that the permutation σ always acts as the identity on roots.

We compute the functor G^{-1} , the inverse of the mutation μ_w^+ centered at w .

$$\begin{aligned} G^{-1} \begin{pmatrix} P_s'' \end{pmatrix} : & \quad 0 \rightarrow P_s' \rightarrow 0, s \neq w \\ G^{-1} \begin{pmatrix} P_w'' \end{pmatrix} : & \quad 0 \rightarrow P_w' \rightarrow P_u' \oplus P_t' \rightarrow 0 \end{aligned}$$

Since w and u are both roots, we have

$$\begin{aligned} F_{\ell-1}^{-1} \begin{pmatrix} P_u' \end{pmatrix} &= H_{\ell-1}^{-1} \begin{pmatrix} P_u' \end{pmatrix} : & 0 \rightarrow P_u \rightarrow 0 \\ F_{\ell-1}^{-1} \begin{pmatrix} P_w' \end{pmatrix} &= H_{\ell-1}^{-1} \begin{pmatrix} P_w' \end{pmatrix} : & 0 \rightarrow P_w \rightarrow 0 \end{aligned}$$

It remains to calculate $F_{\ell-1}^{-1} \begin{pmatrix} P_t' \end{pmatrix}$. By the definition of the alternating pointing, we get $\sigma_{\ell-1}(t) = i$, where, i is the lowest number in the branch rooted at w , and by the definition of the star-to-tree tilting complex of a given numbering, we have

$$F_{\ell-1}^{-1} \begin{pmatrix} P_t' \end{pmatrix} = 0 \rightarrow P_i \rightarrow P_w \rightarrow 0$$

We now calculate F_{ℓ}^{-1} as the composition $F_{\ell-1}^{-1} \circ G^{-1}$. We first make a general claim that if $i \rightarrow j \rightarrow k$ is short, then $P_j \oplus P_i \rightarrow P_k \oplus P_j$ is homotopy equivalent to $T_{ik}[-1]$, where the map is given by $(-h_{jk} \circ \pi_j, \pi_j + h_{ij} \circ \pi_i)$. The chain maps are obvious and the composition from T_{ik} to itself is the identity, so we need only find a homotopy from the opposite composition to the identity:

$$\begin{array}{ccc} P_j \oplus P_i & \xrightarrow{(-h_{jk} \circ \pi_j, \pi_j + h_{ij} \circ \pi_i)} & P_k \oplus P_j \\ \downarrow \pi_i & & \downarrow \pi_k + h_{jk} \circ \pi_j \\ P_i & \xrightarrow{h_{ik}} & P_k \\ \downarrow (-h_{ij}, id) & & \downarrow (id, 0) \\ P_j \oplus P_i & \xrightarrow{(-h_{jk} \circ \pi_j, \pi_j + h_{ij} \circ \pi_i)} & P_k \oplus P_j \end{array}$$

The needed homotopy is given by $T = (-\pi_j, 0)$. With this result in hand, and noting that $i \rightarrow w \rightarrow u$ is short because $i \leq t < w < u$, we make our calculation.

$$\begin{aligned}
F_{l-1}^{-1} \circ G^{-1} (P_w'') &= F_{l-1}^{-1} \left(\text{Cone} \left(P_w' \rightarrow P_u' \oplus P_t' \right) \right) \\
&= \text{Cone} \left(F_{l-1}^{-1}(P_w') \rightarrow F_{l-1}^{-1}(P_u') \oplus F_{l-1}^{-1}(P_t') \right) \\
&= \text{Cone} \left(\begin{array}{ccccc} 0 & \longrightarrow & P_w & \longrightarrow & 0 \\ & & \downarrow (h_{wu}, id) & & \\ P_t & \xrightarrow{(0, h_{tw})} & P_u \oplus P_w & \longrightarrow & 0 \end{array} \right) \\
&= P_w \oplus P_i \rightarrow P_u \oplus P_w \\
&\equiv P_i \xrightarrow{h_{iu}} P_u
\end{aligned}$$

The resulting star-to-tree complex is clearly in TC_2 . Because G^{-1} is almost everywhere trivial, the only differences between F_ℓ and $F_{\ell-1}$ are:

$$\begin{aligned}
F_{\ell-1}^{-1} (P_w') &= H_{\ell-1}^{-1} (Q_w') = & 0 & \rightarrow & P_w & \rightarrow & 0 \\
F_\ell^{-1} (P_w'') &= H_\ell^{-1} (Q_i'') = & 0 & \rightarrow & P_i & \rightarrow & P_u & \rightarrow & 0 \\
F_{\ell-1}^{-1} (P_t') &= H_{\ell-1}^{-1} (Q_i') = & 0 & \rightarrow & P_i & \rightarrow & P_w & \rightarrow & 0 \\
F_\ell^{-1} (P_t'') &= H_\ell^{-1} (Q_w'') = & 0 & \rightarrow & P_i & \rightarrow & P_w & \rightarrow & 0
\end{aligned}$$

Thus σ_ℓ is identical with $\sigma_{\ell-1}$ except on w and t . We have $\sigma_\ell(w) = \sigma_{\ell-1}(t) = i$ and $\sigma_\ell(t) = \sigma_{\ell-1}(w) = w$. It remains only to show the H_ℓ is indeed the star-to-tree functor for the left alternating pointing.

In $B_{\ell-1}$, let T be the collection of branches at the far end of t , let W be the remaining branches connected to w , and let U be the collection of branches at the far end of u . The interval in the natural numbering corresponding to T is $[i, \dots, t-1]$, the interval corresponding to W is $[t+1, \dots, w-1]$, and the interval corresponding to U is $[w+1, \dots, u-1]$. In B_ℓ , w is attached to u before U , t has become coprimary, with W attached to its far end, and T is now attached directly to w . The distances of U and of T from the exceptional vertex v remain as they were, and the distance of W is increased by two, since it was originally

attached directly to w , and now u , w and t intervene. Since the intervals remain the same, and the alternating pointing remains the same, the left alternating numbering for each is the same, and thus σ_ℓ and $\sigma_{\ell-1}$ are identical on U , W and T , and also on u , where both are fixed.

Thus, for every vertex except w and t , the left alternating pointing is exactly as it was. At the far end of w , the point is on the right, and thus the left alternating pointing assigns to w the lowest number in the united branches, which was the lowest number in the branch with root w , which we called i . In the united tree B_ℓ , the edge t is coprimary going out from w and has the point between it and w . Since the edge w has been assigned a low number, the alternating numbering assigns to t the largest number in the branch originally rooted at w , which is w . The alternating numbering then gives to every other edge exactly the same number as before. This proves 1.

2. By [RS] the Rickard tree-to-star complex for the left alternating pointing is the inverse of the star-to-tree complex for the same pointing. Since the star-to-tree for Aihara's algorithm differs from the star-to-tree for the left alternating pointing only in the order of the components, the Aihara complex differs from the Rickard complex only by the same permutation of the projectives.

□

Corollary 5.0.1. *The permutation σ is given by the cyclic ordering of edges on the vertices at non-zero even distance from the exceptional vertex.*

Proof. As in the proof of the proposition, we do an induction on ℓ , assuming that the result holds for $\ell - 1$. Thus in $\sigma_{\ell-1}$, the edges at the far end of t are permuted according to the cyclic ordering at the vertex, from the primary edge i through the numerically increasing starting edges of the branches in T , and then to t , and finally from t back to i . In B_ℓ , this vertex now has an extra edge. The cyclic ordering goes from i through the same sequence of starting edges in T , to the coprimary edge t , and finally to w . This is precisely the change we documented in σ_ℓ , where now w goes to i and t to w , increasing the length of the cycle by one.

□

Example 4. We illustrate the above Proposition with a simple example, which will also demonstrate that the numbering we get from Proposition 5.1 will not, in general, be the natural numbering.

In Figure 10 we have a linear Brauer tree, which we reduce using Aihara's Algorithm to a Brauer star with $e = 5$.

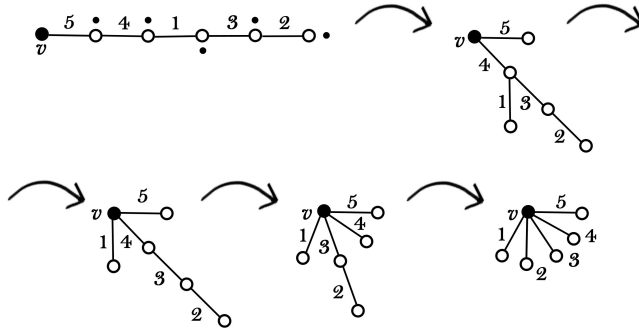


Figure 10

Now we compare this result with the composition of mutations as in Proposition 5.1:

$$\begin{aligned}
 F_4^{-1}(P_1'') : & \quad 0 \rightarrow P_1 \rightarrow P_4 \rightarrow 0 \\
 F_4^{-1}(P_2'') : & \quad 0 \rightarrow P_2 \rightarrow P_3 \rightarrow 0 \\
 F_4^{-1}(P_3'') : & \quad 0 \rightarrow P_2 \rightarrow P_4 \rightarrow 0 \\
 F_4^{-1}(P_4'') : & \quad 0 \rightarrow P_1 \rightarrow P_5 \rightarrow 0 \\
 F_4^{-1}(P_5'') : & \quad \quad \quad 0 \rightarrow P_5 \rightarrow 0
 \end{aligned}$$

This differs from the folded star-to-tree complex for the pointed Brauer tree in Figure 11, constructed as in [RS] by the ordering of the components of the tilting complex. The image of P_1'' and P_4'' are exchanged, as are the images of P_2'' and P_3'' .

$$\begin{aligned}
 H_4^{-1}(Q_1'') : & \quad 0 \rightarrow P_1 \rightarrow P_5 \rightarrow 0 \\
 H_4^{-1}(Q_2'') : & \quad 0 \rightarrow P_2 \rightarrow P_4 \rightarrow 0 \\
 H_4^{-1}(Q_3'') : & \quad 0 \rightarrow P_2 \rightarrow P_3 \rightarrow 0 \\
 H_4^{-1}(Q_4'') : & \quad 0 \rightarrow P_1 \rightarrow P_4 \rightarrow 0 \\
 H_4^{-1}(Q_5'') : & \quad \quad \quad 0 \rightarrow P_5 \rightarrow 0
 \end{aligned}$$

For completeness, we give the corresponding Rickard tree-to-star complex.

$$\begin{array}{llllll}
H_4(P_1) : & 0 & \rightarrow & Q_5'' & \rightarrow & Q_1'' \rightarrow 0 \\
H_4(P_2) : & 0 & \rightarrow & Q_5'' \oplus Q_4'' & \rightarrow & Q_1'' \oplus Q_2'' \rightarrow 0 \\
H_4(P_3) : & 0 & \rightarrow & Q_5'' \oplus Q_4'' \oplus Q_3'' & \rightarrow & Q_1'' \oplus Q_2'' \rightarrow 0 \\
H_4(P_4) : & 0 & \rightarrow & Q_5'' \oplus Q_4'' & \rightarrow & Q_1'' \rightarrow 0 \\
H_4(P_5) : & 0 & \rightarrow & Q_5'' & \rightarrow & 0 \\
\\
F_4(P_1) : & 0 & \rightarrow & P_5'' & \rightarrow & P_4'' \rightarrow 0 \\
F_4(P_2) : & 0 & \rightarrow & P_5'' \oplus P_1'' & \rightarrow & P_4'' \oplus P_3'' \rightarrow 0 \\
F_4(P_3) : & 0 & \rightarrow & P_5'' \oplus P_1'' \oplus P_2'' & \rightarrow & P_4'' \oplus P_3'' \rightarrow 0 \\
F_4(P_4) : & 0 & \rightarrow & P_5'' \oplus P_1'' & \rightarrow & P_4'' \rightarrow 0 \\
F_4(P_5) : & 0 & \rightarrow & P_5'' & \rightarrow & 0
\end{array}$$

The functor can be obtained from the Rickard tree-to-star given by the functor H_4 above by a permutation of projectives exchanging 1 with 4 and of 2 with 3.

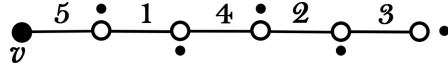


Figure 11

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